On Period-tripling and Quadrupling Bifurcations

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Abstract

This note gives sufficient conditions for period-tripling and period-quadrupling bifurcations for a complex map.

Keywords

Period-tripling Bifurcation; Period-quadrupling Bifurcation

Introduction

In this note, sufficient conditions are available for period-tripling and period-quadrupling bifurcations for a complex map. Such bifurcations do occur in practice but are not as well studied as the more ubiquitous period-doubling bifurcations.

Consider the map $f: C \times R \to C$ taking $(z, \lambda) \to f(z, \lambda)$. Here, λ is a parameter of the problem. For a fixed value of λ, it is recalled that z^* is a fixed point of f if $z^* = f(z^*, \lambda)$. It is said that z^* is a period-2 fixed point if $z^* = f(f(z^*, \lambda), \lambda)$ with z^* not a fixed point of f. Similarly, for any positive integer n, z^* is a period-n fixed point if $z^* = f^n(z^*, \lambda)$ and z^* is not a period-j fixed point for any positive integer $j < n$. Here $f^1(z^*, \lambda) = f(z^*, \lambda)$ and $f^{j+1}(z^*, \lambda) = f(f^j(z^*, \lambda), \lambda)$ for any positive integer j.

For a fixed λ, a period-n fixed point z^* of the map f is said to be stable if $|(f^n)_z(z^*, \lambda)| < 1$ and unstable if $|(f^n)_z(z^*, \lambda)| > 1$. Here, the subscript z denotes derivative with respect to z. If z^* is a stable fixed point of f, then for all z_0 sufficiently close to z^*, we have $f^k(z_0, \lambda) \to z^*$ as $k \to \infty$ with λ fixed.

Under certain conditions, a fixed point of the map undergoes a period-doubling bifurcation. That is, beyond a critical value of λ, the fixed point loses stability and a new stable period-2 fixed point appears. Refer to Devaney, Kuznetsov and Wiggins for three among many textbooks on dynamical systems/bifurcation theory in which these concepts are discussed.

A natural question is under what conditions period-tripling, period-quadrupling and higher-period bifurcations will appear. These phenomena have been reported in the literature. Take a reference to, for instance, Golberg, Sinai and Khanin; Koswami; Hamakiotes and Berger; Isaeva, Kuznetsov and Osbaldestin; Jiang, Perlin and Schultz; Ngai and Liu; and Sieber and Schomerus. However, there is no available work which gives explicit sufficient conditions for these bifurcations. It is the purpose of this short note to address this deficiency. We shall give conditions for period-3 and period-4 bifurcations. In principle, our method can be applied to period-n bifurcations for any positive n.

Period-tripling Bifurcation

Suppose z=0 is a fixed point of a map for all values of the parameter λ. Conditions are given under which period-tripling bifurcation occurs at λ=0. This means that the branch of trivial fixed points loses stability at λ=0 and a branch of stable period-3 fixed points appears when λ>0. See Fig. 1 for an example.

Let $w = e^{i\,2\pi/3}$ or $w = e^{i\,4\pi/3}$ be a third root of unity: $w^3 = 1$.

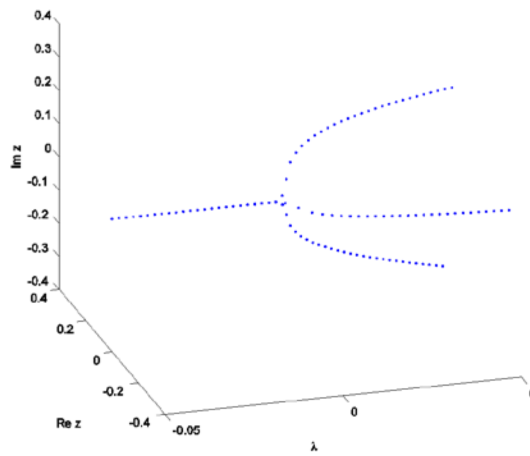

FIG 1 PERIOD-TRIPLING BIFURCATION FOR THE MAP $f(z, \lambda) = z((1 + \lambda)\, w + z)$, WHERE $w = e^{i\,2\pi/3}$. ONLY THE STABLE FIXED POINTS ARE SHOWN.

Theorem 1

Let $f : C \times R \to C$ be an analytic map so that $f(0,\lambda) = 0$ for all λ and $f_z^0 = w$, $f_{z\lambda}^0 = w$ and $\alpha \neq 0$, where $\alpha = a^3(5+w) + ab\,(8 + 7\,w^2)$, $a = f_{zz}^0/2$, $b = f_{zzz}^0/6$. Here, the superscript ^{0} denotes evaluation at the point$(z,\lambda) = (0,0)$. Then there is a period-tripling bifurcation at (0,0). The fixed point zero is stable when $\lambda < 0$ and unstable when $\lambda > 0$. The period-3 fixed points are unstable when $\lambda < 0$ and stable when $\lambda > 0$. These stability results hold for all λ in some neighbourhood of the origin.

Proof:

From the assumptions on f, we can write

$$f(z,\lambda) = z\,w + z\,\lambda\,w + a\,z^2 + b\,z^3 + O(\lambda z^2, z^4).$$

(We have omitted terms of order $\lambda^2\,z$ because, as we shall see, they are insignificant compared to $O(\lambda z^2, z^4)$). To find a period-tripling bifurcation, we seek the zeroes of

$$g(z,\lambda) := f(f(f(z,\lambda),\lambda),\lambda) - z.$$

Using Maple, we find the expansion

$$\begin{aligned} g(z,\lambda) = {} & 3\,\lambda\,z + a(1 + w + w^2)\,z^2 + 2\,(a^2 + b)\,(1 + w + w^2)\,z^3 + \\ & (a^3\,(6 + 2w + w^2) + ab\,(8 + 7\,w^2))\,z^4 + O(\lambda z^2, z^5) \\ = {} & 3\,\lambda\,z + \alpha\,z^4 + O(\lambda z^2, z^5), \end{aligned}$$

where the equality $1 + w + w^2 = 0$ has been used. Hence the leading-order term of the nontrivial zeroes ζ of g satisfies $3\,\lambda + \alpha\,\zeta^3 = 0$. Since this is a cubic equation, there are exactly three zeroes $\zeta = \zeta_j(\lambda)$, $j = 1,2,3$ with $\zeta_j(0) = 0$. It is demonstrated that they correspond to three branches of nontrivial solutions of $g = 0$: $(z_j(\lambda),\lambda)$ for all λ in some neighbourhood of 0 with $z_j(0) = 0$. Consider the near-identity transformation φ_j mapping the nontrivial solution of

$$\widetilde{g}\,(\zeta_j) := 3\,\lambda\zeta_j + \alpha\,\zeta_j^4 = 0$$

to that of $g(z_j) = 0$:

$$z_j = \varphi(\zeta_j) := \zeta_j + r_j\zeta_j^2$$

for some r_j to be determined. Let

$$g(z,\lambda) = 3\,\lambda\,z + \alpha\,z^4 + p\,\lambda z^2 + q\,z^5 + O(\lambda z^3, z^6),$$

for some complex constants p and q. Now

$$\begin{aligned} 0 &= g(z_j,\lambda) \\ &= 3\,\lambda(\zeta_j + r_j\zeta_j^2) + \alpha\,(\zeta_j + r_j\zeta_j^2)^4 + p\,\lambda(\zeta_j + r_j\zeta_j^2)^2 + q\,(\zeta_j + r_j\zeta_j^2)^5 + O(\lambda\zeta_j^3, \zeta_j^6) \\ &= -\alpha\,\zeta_j^4 - \alpha\,r_j\,\zeta_j^5 + \alpha\,\zeta_j^4 + 4\,\alpha\,r_j\,\zeta_j^5 - \frac{p\,\alpha}{3}\zeta_j^5 + q\,\zeta_j^5 + O(\zeta_j^6) \\ &= \left(3\,\alpha\,r_j - \frac{p\,\alpha}{3} + q\right)\zeta_j^5 + O(\zeta_j^6). \end{aligned}$$

Since nontrivial solutions are the focus, the following is

$$\begin{aligned} 0 = G(\zeta_j, r_j) &:= \frac{g(z_j,\,\lambda)}{\zeta_j^5} \\ &= \left(3\alpha\,r_j - \frac{p\,\alpha}{3} + q\right) + O(\zeta_j). \end{aligned}$$

Let

$$r_j^* = \frac{p}{9} - \frac{q}{3\alpha}.$$

Note that $G^* = 0$ and $G_{r_j}^* = 3\,\alpha \neq 0$, where * denotes evaluation at the point $(\zeta_j, r_j) = (0, r_j^*)$.Hence by means of the implicit function theorem, there is some analytic function $R_j = R_j(\zeta_j)$ so that $G(\zeta_j, R_j\,(\zeta_j)) = 0$, for all ζ_j in some neighbourhood of 0 with $R_j(0) = r_j^*$. In summary, for all λ in some neighbourhood of 0,

$$z_j(\lambda) = \varphi_j(\zeta_j(\lambda)) = \zeta_j(\lambda) + R_j(\zeta_j(\lambda))\,\zeta_j^2(\lambda)$$

satisfies

$$g(z_j(\lambda),\lambda) = 0.$$

From the form of φ_j, for all ζ_j in some neighbourhood of the origin, the inverse of φ_j exists. Thus

$$\widetilde{g}\,(\varphi_j^{-1}(z_j(\lambda)),\lambda) = \widetilde{g}\,(\zeta_j(\lambda),\lambda) = 0$$

for all λ in some neighbourhood of 0. This completes the proof that solutions of $\widetilde{g} = 0$ are exactly those of $g = 0$ in a small neighbourhood of $\lambda = 0$. Since $\alpha \neq 0$ and $3\,\lambda = -\alpha\,\zeta_j^3$, the point $z_j(\lambda)$ can be neither a fixed point nor a period-2 fixed point of f. Thus $z_j(\lambda)$ must be a period-3 fixed point of f.

Next, stability of the fixed points to f will be investigated. For the zero fixed point, $|f_z(0,\lambda)| = |(1 + \lambda)\,w| = |1 + \lambda|$. Thus the zero fixed point is stable if $-2 < \lambda < 0$ and unstable if $\lambda > 0$. For the period-3 fixed points,

$$\begin{aligned} (f^3)_z(z_j(\lambda),\lambda) &= 1 + g_z(z_j(\lambda),\lambda) \\ &= 1 + 3\lambda + 4\,\alpha\,z_j^3(\lambda) + O(\lambda z_j, z_j^4) \\ &= 1 - 9\lambda + o(\lambda). \end{aligned}$$

Consequently, the period-3 fixed points are stable when $\lambda > 0$and unstable when $\lambda < 0$ for $|\lambda|$ sufficiently small.

It is easy to transplant the above result to 2D real maps $(x,y) \to (g(x,y), h(x,y))$.Let $z = x + iy$ and $f = g + ih$. If g and h are continuously differentiable functions and satisfy the Cauchy-Riemann equations, then it is known that f is analytic. Using the Cauchy-Riemann equations, it is easy to check that

$$f_z = g_x - ig_y, \quad f_{z\lambda} = g_{x\lambda} - ig_{y\lambda},$$
$$f_{zz} = g_{xx} - ig_{xy},$$
$$f_{zzz} = g_{xxx} - ig_{xxy}.$$

Corollary 2

Let $g, h : R^3 \to R$ be real three times continuously differentiable functions that satisfy the Cauchy-Riemann equations. Assume

$g(0,0,\lambda) = 0 = h(0,0,\lambda)$ for all λ and $g_x^0 - i\, g_y^0 = w$, $g_{x\lambda}^0 - i\, g_{y\lambda}^0 = w$ and $\alpha \neq 0$,

where $\alpha = a^3\,(5+w) + ab(8 + 7\,w^2)$, $a = (g_{xx}^0 - i\, g_{xy}^0)/2$, $b = (g_{xxx}^0 - ig_{xxy}^0)/6$. Here, the superscript ^{0}denotes evaluation at the point $(x, y, \lambda) = (0,0,0)$. Then the map $(x, y) \to (g(x, y), h(x, y))$ undergoes a period-tripling bifurcation at (0,0,0). The fixed point zero is stable when $\lambda < 0$and unstable when $\lambda > 0$. The period-3 fixed points are unstable when lambda < 0 and stable when lambda > 0. These stability results hold for all λ in some neighbourhood of the origin.

As an example, consider $g(x, y) = (1+\lambda)\,(Re\ w)\ x - (1+\lambda)\,(Im\ w)\ y + x^2 - y^2$ and $h(x, y) = (1+\lambda)\,(Im\ w)\ x + (1+\lambda)\,(Re\ w)\ y + 2xy$. The corresponding complex map is $f(z) = z\,((1+\lambda)w + z)$.

Period-quadrupling Bifurcation

Suppose $z = 0$ is a fixed point of a map for all values of the parameter λ. Then attention has been shifted to conditions giving rise to period- quadrupling bifurcation at $\lambda = 0$. That is, the branch of trivial fixed points loses stability at $\lambda = 0$ and a branch of stable period-4 fixed points emerges when $\lambda > 0$. See Fig. 2 for an example.

Let $w = \pm\, i$ be a fourth root of unity: $w^4 = 1$.

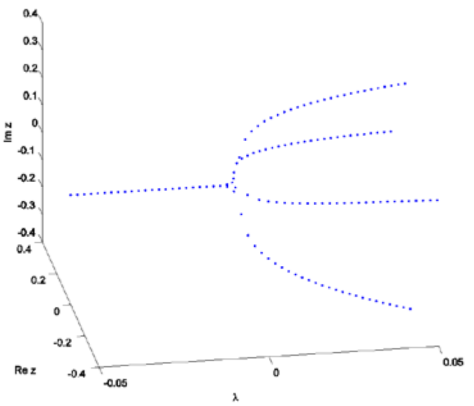

FIG 2. PERIOD-QUADRUPLING BIFURCATION FOR THE MAP $f(z,\lambda) = z((1+\lambda)\, i\, +\, z)$. ONLY THE STABLE FIXED POINTS ARE SHOWN.

Theorem 3

Let $f : C \times R \to C$ be an analytic map so that $f(0,\lambda) = 0$ for all λ and $f_z^0 = w$, $f_{z\lambda}^0 = w$ and $\alpha \neq 0$,where$\alpha = 2a^2 b\,(-5+12w) - 2a^4(5+3w) + 6\,b^2$, $a = f_{zz}^0/2$, $b = f_{zzz}^0/6$. Here, the superscript ^{0}denotes evaluation at the point $(z, \lambda) = (0,0)$. Then there is a period-quadrupling bifurcation at (0,0). The fixed point zero is stable when $\lambda < 0$ and unstable when$\lambda > 0$. The period-4 fixed points are unstable when $\lambda < 0$ and stable when$\lambda > 0$. These stability results hold for all λ in some neighbourhood of the origin.

Proof

The proof is very similar to the period-tripling case and the main points are the priority. From the assumptions on f, we can write

$$f(z,\lambda) = z\,w + z\,\lambda\,w + \alpha\,z^2 + b\,z^3 + O(\lambda z^2, z^4).$$

To find a period-quadrupling bifurcation, we seek the zeroes of

$$g(z,\lambda) = f(f(f(f(z,\lambda),\lambda),\lambda),\lambda) - z.$$

it is observed that $w^2 + 1 = (\pm\, i)^2 + 1 = 0$. Using Maple, the expansion is found to be,

$$g(z,\lambda) = 4\,\lambda\,z + \alpha\,z^5 + O(\lambda z^2, z^6).$$

Hence the leading-order term of the nontrivial zeroes $\zeta$of g satisfies$4\,\lambda + \alpha\,\zeta^4 = 0$. Since this is a quartic equation, there are exactly four zeroes $\zeta = \zeta_j(\lambda)$, $j = 1,2,3,4$ with $\zeta_j(0) = 0$. Using the near-identity transformation $z_j = \varphi(\zeta_j) = \zeta_j + r_j\zeta_j^2$, it can be shown that they correspond to four nontrivial branches of solutions of $g = 0$: $(z_j(\lambda), \lambda)$for all λ in some neighbourhood of 0 with$z_j(0) = 0$. They are period-4 fixed points of$f$.

Now the stability of the period-4 fixed points are checked. Note that

$$\begin{aligned}(f^4)_z(z_j(\lambda),\lambda) &= 1 + g_z(z_j(\lambda),\lambda)\\ &= 1 + 4\,\lambda + 5\,\alpha\,z_j^4(\lambda) + O(\lambda z_j, z_j^5)\\ &= 1 - 16\,\lambda + o(\lambda).\end{aligned}$$

Consequently, the period-4 fixed points are stable when $\lambda > 0$ and unstable when $\lambda < 0$ for all $|\lambda|$ sufficiently small.

Corollary 4

Let $g, h : R^3 \to R$ be real three times continuously differentiable functions that satisfy the Cauchy-Riemann equations. Assume

$g(0,0,\lambda) = 0 = h(0,0,\lambda)$ for all λ and $g_x^0 - i\, g_y^0 = w$, $g_{x\lambda}^0 - i\, g_{y\lambda}^0 = w$ and $\alpha \neq 0$,

where $\alpha = 2a^2 b\,(-5+12w) - 2a^4(5+3w) + 6b^2$,

$a = (g_{xx}^0 - i\, g_{xy}^0)/2$, $b = (g_{xxx}^0 - i g_{xxy}^0)/6$. Here, the superscript ^{0} denotes evaluation at the point $(x, y, \lambda) = (0,0,0)$. Then the map $(x, y) \to (g(x, y), h(x, y))$ undergoes a period-quadrupling bifurcation at $(0,0,0)$. The fixed point zero is stable when $\lambda < 0$ and unstable when $\lambda > 0$. The period-4 fixed points are unstable when $\lambda < 0$ and stable when $\lambda > 0$. These stability results hold for all λ in some neighbourhood of the origin.

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